Optimal universal two-particle entanglement processes in arbitrary dimensional Hilbert spaces

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Universal two-particle entanglement processes are analyzed in arbitrary dimensional Hilbert spaces. On the basis of this analysis the class of possible optimal universal entanglement processes is determined whose resulting output states do not contain any separable states. It is shown that these processes form a one-parameter family. For all Hilbert space dimensions larger than two the resulting optimally entangled output states are mixtures of anti-symmetric states which are freely entangled and which also preserve information about input states. Within this one-parameter family there is only one process by which all information about any input state is destroyed completely.

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I. INTRODUCTION

One of the main driving forces in the rapidly developing field of quantum information processing is the question whether basic quantum phenomena such as interference and entanglement can be exploited for practical purposes. In this context it has been realized that the linear character of quantum theory may impose severe restrictions on the performance of elementary tasks of quantum information processing. As a consequence it is impossible to copy (or clone) an arbitrary quantum state perfectly [1]. In view of the significance of entangled states for many aspects of quantum information processing the natural question arises whether similar restrictions also hold for quantum mechanical entanglement processes. Of particular interest are entanglement processes which entangle two quantum systems in an optimal way in the sense that the corresponding two-particle quantum state does not contain any separable components. Though many quantum mechanical processes are capable of entangling some input states of a quantum system with a known reference state of a second quantum system, it is not easy to achieve this goal for all possible input states. This basic difficulty can be realized already in the simple example of a quantum mechanical controlled-not (CNOT) operation, i.e.

$$\mathrm{CNOT} : |\pm\rangle \otimes (|+\rangle + |-\rangle) \to |\mp\rangle \otimes |+\rangle + |\pm\rangle \otimes |-\rangle.$$

This CNOT operation entangles the orthogonal input states $|\pm\rangle$ of the first qubit with the second (control) qubit prepared in the reference state $(|+\rangle + |-\rangle)$. Obviously the two Bell states resulting from these input states are optimally entangled. However, due to its linearity this quantum process is incapable of entangling the first qubit with the second one for all possible input states. The input state $(|+\rangle + |-\rangle)$, for example, results in the factorizable output state $(|+\rangle + |-\rangle) \otimes (|+\rangle + |-\rangle)$. In view of this difficulty it is of particular interest to investigate universal entanglement processes which are able to entangle all possible input states of a quantum system with a second one in an optimal way.

Universal quantum processes act on all possible (typically pure) input states of a quantum system in a 'similar' way. Consequently, these processes do not specify a preferred direction in Hilbert space and thus reflect its 'natural' symmetry. Therefore, the restrictions imposed on these processes by the linear character of quantum theory are not only of practical interest but they also hint at fundamental limits of quantum theory. So far many properties of universal quantum processes have been analyzed for qubits [2,3]. For qubits one can show that there is only one universal, optimal entanglement process. Independent of the input states this process produces the anti-symmetric Bell state as an optimally entangled output state [4]. However, for many applications in quantum information processing universal quantum processes are needed which do not only entangle different quantum systems in an optimal way but which also preserve information about the original input state and redistribute this information into an entangled two-particle state. Motivated by this need recently Bužek and Hillery [3] have analyzed various quantum processes which entangle two qubits and which also preserve information about the initial input state. Though for universal quantum processes both requirements are incompatible in the case of qubits universal optimal cloning processes manage to optimize both tasks simultaneously. However, the resulting output states are not optimally entangled as they always contain a separable two-qubit state. From these investigations on qubit systems one may be tempted to presume that a similar incompatibility between optimal entanglement and preservation of information about input states also holds in higher dimensional Hilbert spaces.

In this paper it is shown that, contrary to this tempting presumption, in Hilbert spaces of dimensions higher than two optimal universal entanglement processes are possi-

ble which preserve information about the initial input state and which also lead to optimally entangled output states. For this purpose a convenient theoretical framework is developed which is capable of describing all possible universal quantum processes involving two quantum systems. For the sake of simplicity we restrict our discussion to the important special case that the dimensions of the Hilbert spaces of both quantum systems are equal. First of all, the general class of all possible universal quantum operations is determined which is compatible with the linear character of quantum mechanics. Secondly, the particular subclass is determined which produces optimally entangled two-particle output states in the sense that these output states do not contain any separable components. It turns out that for Hilbert spaces with dimensions larger than two these optimal universal entanglement processes form a one-parameter family. Within this one-parameter family there is only one particular process producing optimally entangled output states which are independent of the input states. This particular optimal universal entanglement process has already been discussed previously [4]. All other processes within this one-parameter family lead to output states which also contain information about the input state. One of these processes preserves this information about the original input state in an optimal way. It turns out that all the resulting optimally entangled output states are anti-symmetric with respect to particle exchange.

This paper is organized as follows: In Sec. II the basic symmetry (or covariance) property of universal quantum processes is discussed by starting from a simple example. subsequently a general formalism is developed for describing all possible universal quantum processes in arbitrary dimensional Hilbert spaces. The consequences of covariance and of the linear character of universal quantum processes are implemented. In Sec. III all possible optimal universal entanglement processes are determined and basic properties of the resulting output states are investigated.

II. UNIVERSAL QUANTUM PROCESSES

In this section the basic symmetry property of universal quantum processes is exemplified by considering a simple example involving a process with two qubits. Based on this symmetry property and on the requirement that any quantum process has to be linear with respect to all possible input states the general structure of universal quantum processes is discussed for the case of two arbitrary dimensional quantum systems. Optimal universal quantum cloning processes and optimal universal entanglement processes are special cases thereof.

A. Symmetry of universal quantum processes - an example

Let us consider the following quantum process as an introductory example:
Initially we prepare two distinguishable spin-1/2 quantum systems (qubits) in the state

$$\rho_1(\mathbf{m}) \equiv \rho_{in}(\mathbf{m}) \otimes \frac{1}{2}\mathbf{1}.$$

The pure input state $\rho_{in}(\mathbf{m}) = |\mathbf{m}\rangle\langle\mathbf{m}|$ of the first quantum system can be described by its Bloch vector $\mathbf{m}$. This Bloch vector can take an arbitrary position on the Poincare sphere. The second quantum system is in a completely unpolarized mixed reference state which is assumed to be fixed once and for all. Selecting an arbitrary pure input state $\rho_{in}(\mathbf{m})$ we transfer the initial state $\rho_1(\mathbf{m})$ into the output state

$$\rho_1(\mathbf{m}) \to \rho_2(\mathbf{m}) = \frac{\mathbf{P}_J\rho_1(\mathbf{m})\mathbf{P}_J}{\mathrm{Tr}\mathbf{P}_J\rho_1(\mathbf{m})\mathbf{P}_J}. \quad (1)$$

Thereby the projection operator $\mathbf{P}_J = \sum_M |JM\rangle\langle JM|$ projects onto two-particle states with well defined total angular momentum J. This total angular momentum can assume the possible values $J = 1$ or $J = 0$. In a probabilistic way the transformation of Eq.(1) can be achieved by a measurement process. However, one may also think of realizing this transformation with a probability of unity in a unitary and reversible way, for example. Choosing the direction of polarization of the input state as the quantization axis the result of this quantum process is given by

$$\rho_2(\mathbf{m}) = p_1|J=1M=1\rangle\langle J=1M=1| + (1-p_1)|J=1M=0\rangle\langle J=1M=0| \quad (2)$$

with $p_1 = 2/3$ or by

$$\rho_2(\mathbf{m}) = |J=0M=0\rangle\langle J=0M=0| \quad (3)$$

depending on whether $J = 1$ or $J = 0$. This quantum process is universal in the sense that all input states are treated in a 'similar' way. The only direction the output state depends on is the one of the input state. Thus this quantum process is symmetric with respect to unitary transformations U which transform an arbitrary pure one-particle input state, say $|\mathbf{m}_0\rangle$, into some other pure one-particle input state, say $|\mathbf{m}\rangle \equiv U|\mathbf{m}_0\rangle$. This unitary symmetry or covariance of such a universal quantum process is characterized by the relation

$$\rho_2(\mathbf{m}) = U(\mathbf{m}) \otimes U(\mathbf{m})\rho_2(\mathbf{m}_0)U^\dagger(\mathbf{m}) \otimes U^\dagger(\mathbf{m}) \quad (4)$$

(compare with Fig. 1). Thus the possible output states of a universal quantum process constitute a two-particle

representation of the group of unitary one-particle transformations.

Universal quantum processes in which the step of Eq.(1) can be implemented with a probability of unity have been investigated in the context of copying (cloning) quantum states. In particular, it has been demonstrated that optimal quantum cloning can be achieved always by a universal quantum process. Furthermore, in the case of two qubits the maximum probability with which an optimal universal quantum cloning process is successful is given by $2/3$ [2]. This latter probability is identical with the probability p_1 appearing in Eq.(2). Thus, provided the process of Eq.(1) is implemented for $J = 1$ and with a probability of unity this process copies an arbitrary input state in an optimal way. However, if we project onto states with $J = 0$, we end up in the anti-symmetric Bell state formed by both qubits. Furthermore, this output state is independent of the input state which we choose. As a Bell state is maximally entangled this latter type of process is an example of a universal optimal entanglement process.

Copying quantum states and preparing entangled quantum states are elementary tasks of quantum information processing. Thereby universal quantum processes fulfilling Eq.(4) which exhibit the same symmetry as the set of all possible pure one-particle input states are of special interest. Though much is already known about universal quantum cloning processes almost nothing is known about universal quantum processes which yield optimally entangled quantum states, in particular in arbitrary dimensional Hilbert spaces. The main questions which will be addressed in the following are: Is it possible at all to produce optimally entangled quantum states by universal quantum processes? Which limitations are imposed on the structure of these states by the universality and linearity of such a quantum process? How do the properties of resulting optimally entangled states depend on the dimensionality of the Hilbert spaces involved?

B. General structure of universal quantum processes involving two quantum systems

Let us consider the most general universal quantum process which is capable of entangling two quantum systems, i.e.

$$\mathcal{P} : \rho_{in}(\mathbf{m}) \otimes \rho_{ref} \to \rho_{out}(\mathbf{m}). \tag{5}$$

In our previous example the fixed reference state ρ_{ref} was maximally mixed. In the present case we leave its form unspecified. The density operator of the pure input state is denoted $\rho_{in}(\mathbf{m})$. For the sake of simplicity let us assume that the dimensions of the Hilbert spaces for both quantum systems are equal and of magnitude D. In order to classify all possible universal quantum processes of the form of Eq.(5) we have to determine the most general form of the input and output states.

The density operator of an arbitrary input state of a D dimensional quantum system can always be represented in terms of the generators $\mathbf{A}_{ij}$ $(i, j = 1, ..., D)$ of the group SU_D, i.e.

$$\rho_{in}(\mathbf{m}) = \frac{1}{D}(\mathbf{1} + m_{ij}\mathbf{A}_{ij}). \tag{6}$$

(We use the Einstein summation convention in which one has to sum over all indices which appear in an expression twice.) A representation of these generators is given by the $D \times D$ matrices

$$(\mathbf{A}_{ij})^{(kl)} = \delta_{ik}\delta_{jl} - \frac{1}{D}\delta_{ij}\delta_{kl}. \tag{7}$$

These matrices are not hermitian but they fulfill the relation $\mathbf{A}_{ij}^{\dagger} = \mathbf{A}_{ji}$. Due to the constraint $\sum_{i=1}^{D} \mathbf{A}_{ii} = 0$ only $(D^2 - 1)$ of them are linearly independent. For D=2 these matrices reduce to the well known spherical components of the Pauli spin matrices, i.e. $2\mathbf{A}_{11} = \sigma_z$, $2\mathbf{A}_{12} = \sigma_x + i\sigma_y$ and $2\mathbf{A}_{21} = \sigma_x - i\sigma_y$. Furthermore, $\rho_{in}(\mathbf{m}) = \rho_{in}(\mathbf{m})^{\dagger}$ implies the relations $[m_{ij}]^* = m_{ji}$ so that Eq.(6) involves $(D^2 - 1)$ real-valued and linearly independent parameters m_{ij} which form the components of a generalized Bloch vector. This Bloch vector is an observable of the quantum system which defines its state uniquely. Eq.(6) represents a pure state provided $\mathrm{Tr}(\rho_{in}(\mathbf{m})^2) = 1$ which is equivalent to the constraint $\sum_{i,j} |m_{ij}|^2 - (1/D)(\sum_i m_{ii})^2 = D(D-1)$.

In terms of the generators of Eq.(7) the most general two-particle output state is represented by a density operator of the form

$$\begin{aligned}\rho_{out}(\mathbf{m}) = \frac{1}{D^2}\mathbf{1} \otimes \mathbf{1} + \alpha_{ij}^{(1)}(\mathbf{m})\mathbf{A}_{ij} \otimes \mathbf{1} + \\ \alpha_{ij}^{(2)}(\mathbf{m})\mathbf{1} \otimes \mathbf{A}_{ij} + K_{ijkl}(\mathbf{m})\mathbf{A}_{ij} \otimes \mathbf{A}_{kl}.\end{aligned} \tag{8}$$

In order to implement the covariance condition (4) it is useful to separate the last term of Eq.(8) into terms which are invariant and into terms which transform as the generators $\mathbf{A}_{ij}$ under arbitrary unitary transformations of the form $U \otimes U$. For this purpose it is useful to start from the commutation relations of SU_D, namely

$$[\mathbf{A}_{ij}, \mathbf{A}_{mn}] = \mathbf{A}_{ab}(\delta_{jm}\delta_{ai}\delta_{bn} - \delta_{in}\delta_{am}\delta_{bj}). \tag{9}$$

These relations imply that the tensor products $\mathbf{A}_{ji} \otimes \mathbf{A}_{sj}$ transform under arbitrary transformations of the form $U \otimes U$ in the same way as $\mathbf{A}_{si}$ transforms under transformation of the form U. Furthermore, the tensor product $\mathbf{A}_{ij} \otimes \mathbf{A}_{ji}$ is an invariant under arbitrary unitary transformations of the form $U \otimes U$. However, note that the combination $\mathbf{A}_{ij} \otimes \mathbf{A}_{sj}$, for example, does not transform analogous to $\mathbf{A}_{si}$. Using these elementary transformation properties, the covariance condition (4), and the fact that

any quantum operation has to be linear with respect to its input states we obtain the most general form for the density operator of the two-particle output state, namely

$$\rho_{out}(\mathbf{m}) = \frac{1}{D^2}\mathbf{1}\otimes\mathbf{1} + \alpha^{(1)}_{ij}(\mathbf{m})\mathbf{A}_{ij}\otimes\mathbf{1} + \\ +\alpha^{(2)}_{ij}(\mathbf{m})\mathbf{1}\otimes\mathbf{A}_{ij} + C\mathbf{A}_{ij}\otimes\mathbf{A}_{ji} + \\ \beta_{il}(\mathbf{m})\mathbf{A}_{ij}\otimes\mathbf{A}_{jl} + \beta_{il}(\mathbf{m})^*\mathbf{A}_{ji}\otimes\mathbf{A}_{lj} \quad (10)$$

with

$$\alpha^{(1,2)}_{ij} = \alpha^{(1,2)}m_{ij}, \\ \beta_{ij} = \beta m_{ij} \quad (11)$$

and with C being independent of $\mathbf{m}$.

So far the output state of Eq.(10) represents the most general hermitian operator which depends linearly on the input state $\rho_{in}(\mathbf{m})$ and which fulfills the covariance condition (4). Accordingly, a particular universal quantum process is characterized by the set of real-valued parameters C, $\alpha^{(1)}$, $\alpha^{(2)}$ and by the complex valued parameter β. We still have to solve the more difficult task to restrict the range of these parameters in such a way that $\rho_{out}(\mathbf{m})$ of Eq.(10) represents a non-negative operator. In order to determine this fundamental range of these parameters we have to investigate the possible eigenvalues of the density operator $\rho_{out}(\mathbf{m})$ of Eq.(10). Due to the covariance condition (4) we may restrict this investigation to a particular pure input state with $m_{ij} = D\delta_{i1}\delta_{j1}$, for example. Using the matrix representations of Eq.(7) it turns out that the corresponding output state can be represented by a direct sum of density operators according to

$$\rho_{out}(\mathbf{m} \equiv D\mathbf{e}_{11}) = \sum_{i=1}^{4} \oplus p_i\rho_i \quad (12)$$

with the partial density operators

$$\rho_1 = |11\rangle\langle 11|, \\ \rho_2 = \sum_{j=2}^{D}\{|1j\rangle\langle 1j|(\frac{1}{2(D-1)} + \frac{(\alpha^{(1)}-\alpha^{(2)})m_{11}}{2p_2}) + \\ |j1\rangle\langle j1|(\frac{1}{2(D-1)} + \frac{(\alpha^{(2)}-\alpha^{(1)})m_{11}}{2p_2}) + \\ |1j\rangle\langle j1|\frac{C+\beta m_{11}}{p_2} + |j1\rangle\langle 1j|\frac{C+\beta^* m_{11}}{p_2}\}, \\ \rho_3 = \frac{1}{(D-1)}\sum_{j=2}^{D}|jj\rangle\langle jj|, \\ \rho_4 = \sum_{2=i<j}^{D}\{|ij\rangle\langle ij|\frac{1}{(D-1)(D-2)} + \\ |ji\rangle\langle ji|\frac{1}{(D-1)(D-2)} + \\ |ij\rangle\langle ji|\frac{C}{p_4} + |ji\rangle\langle ij|\frac{C}{p_4}\}. \quad (13)$$

The corresponding partial probabilities entering Eq.(12) are given by

$$p_1 = \frac{1}{D^2} + (\alpha^{(1)}+\alpha^{(2)})m_{11}(1-\frac{1}{D}) + C(1-\frac{1}{D}) + \\ (\beta+\beta^*)m_{11}(1-\frac{1}{D})^2, \\ p_2 = (D-1)\{\frac{2}{D^2} + (\alpha^{(1)}+\alpha^{(2)})m_{11}(1-\frac{2}{D}) - \\ \frac{2C}{D} - 2(\beta+\beta^*)m_{11}(1-\frac{1}{D})\frac{1}{D}\}, \\ p_3 = (D-1)\{\frac{1}{D^2} - \frac{\alpha^{(1)}m_{11}}{D} - \frac{\alpha^{(2)}m_{11}}{D} + \\ C(1-\frac{1}{D}) + (\beta+\beta^*)m_{11}\frac{1}{D^2}\}, \\ p_4 = (D-1)(D-2)\{\frac{1}{D^2} - \frac{\alpha^{(1)}m_{11}}{D} - \frac{\alpha^{(2)}m_{11}}{D} - \\ \frac{C}{D} + (\beta+\beta^*)m_{11}\frac{1}{D^2}\}. \quad (14)$$

The normalization of the density operator, i.e. $\mathrm{Tr}(\rho_{out}(\mathbf{m})) = 1$, implies

$$p_1 + p_2 + p_3 + p_4 = 1. \quad (15)$$

From Eqs.(13) and (14) one obtains the eigenvalues of $\rho_{out}(\mathbf{m} = D\mathbf{e}_{11})$, namely

$$\lambda_1 = p_1, \\ \lambda_{2\pm} = \frac{p_2}{2(D-1)} \pm \sqrt{(\frac{(\alpha^{(1)}-\alpha^{(2)})m_{11}}{2})^2 + |\ C + m_{11}\beta\ |^2}, \\ \lambda_3 = \frac{p_3}{(D-1)}, \\ \lambda_{4\pm} = \frac{p_4}{(D-1)(D-2)} \pm |\ C\ |\ . \quad (16)$$

Therefore the density operator of Eq.(12) is non-negative only if all probabilities p_i and all eigenvalues λ_i of Eqs.(14) and (16) are non-negative and fulfill Eq.(15). For $\alpha^{(1)} = \alpha^{(2)}$ and $\beta = \beta^*$, for example, these conditions on (p_2, p_3, p_4) form a tetrahedron (compare with Fig.2). Each point in this convex set defines a different universal quantum process whose possible output states can be obtained from Eq.(12) with the help of the covariance condition (4). The universal quantum cloning process, for example, is represented by point B in this figure and it is characterized by this particular universal process which maximized p_1. Note that it is immediately obvious from Fig.2 that perfect quantum cloning is impossible with a universal quantum process as $p_1 = 1 - p_2 - p_3 - p_4 \leq 2/(D+1) < 1$.

Finally, it should be mentioned that for dimensions $D \geq 3$ one may choose the probabilities (p_1, p_3, p_4) or (p_2, p_3, p_4), for example, as independent coordinates instead of the three independent real-valued parameters $((\alpha^{(1)}+\alpha^{(2)}), C, (\beta+\beta^*))$. Inverting Eqs.(14) and using

Eq.(15) one obtains the relation between these different coordinate systems, namely

$$\begin{aligned}
\beta+\beta^* &= -\frac{1}{D(D-1)}+\frac{p_4}{(D-1)(D-2)}+\frac{p_1}{(D-1)},\\
\alpha^{(1)}+\alpha^{(2)} &= \frac{(D-2)}{2D^2(D-1)}-\frac{p_4}{2D(D-1)}+\frac{p_1}{2D^2(D-1)}-\\
&\quad \frac{p_3}{2D(D-1)},\\
C &= \frac{p_3}{(D-1)}-\frac{p_4}{(D-1)(D-2)}.
\end{aligned} \tag{17}$$

In order to identify a particular universal quantum process uniquely in addition to these three probabilities one also has to specify the remaining two independent parameters, namely $(\alpha^{(1)}-\alpha^{(2)})$ and $(\beta-\beta^*)$.

III. OPTIMAL UNIVERSAL ENTANGLEMENT PROCESSES

Starting from the notion of optimal entanglement as defined by Lewenstein and Sanpera [5] it is shown that there is a unique one-parameter family of optimal universal entanglement processes for Hilbert spaces with dimensions larger than two. All these processes produce output states which are anti-symmetric with respect to particle exchange. Characteristic properties of the resulting output states are investigated. It is demonstrated that among this one-parameter family of optimal entanglement processes there is only one process in which all information about input states is lost. All other processes preserve this information at least partly. Within this one-parameter family there is also one particular process which preserves this information optimally.

A. Characterization of optimal universal entanglement processes

Is it possible to entangle two quantum systems in an optimal way by a universal quantum process? Before addressing this question one has to clarify the meaning of optimal entanglement. As discussed by Lewenstein and Sanpera [5] one can decompose any quantum state ρ of a composite system into a separable part, say ρ_{sep}, and an inseparable contribution ρ_{insep}, i.e. $\rho=\lambda\rho_{sep}+(1-\lambda)\rho_{insep}$ with $0\leq\lambda\leq 1$. Thereby a separable state is a convex sum of product states of the form $\rho_A\otimes\rho_B$ where ρ_A and ρ_B refer to quantum systems A and B respectively. Though this decomposition itself is not unique the maximum value of λ is. Thus a quantum state may be called optimally entangled if the maximum possible value of λ equals zero in any such decomposition.

In order to determine the parameters for the universal quantum processes which produce optimally entangled states let us start from the output state $\rho_{out}(\mathbf{m}=D\mathbf{e}_{11})$ of Eq.(12). A necessary condition for this state being optimally entangled is that there are no admixtures of separable states of the form $|jj\rangle\langle jj|$ for any $j=1,...,D$. Thus, necessarily a universal quantum process producing optimally entangled states has to be characterized by the parameters

$$\begin{aligned}
p_1 &= 0,\\
p_3 &= 0.
\end{aligned} \tag{18}$$

It will be demonstrated by the subsequent arguments that this choice of parameters is also sufficient for the generation of optimally entangled output states. For this purpose it has to be proven that for any separable two-particle state $|\psi\rangle=|\varphi\rangle\otimes|\chi\rangle$ and any positive value of $\lambda>0$ the state

$$\rho'=\rho_{out}(\mathbf{m}=D\mathbf{e}_{11})^{(opt)}-\lambda|\psi\rangle\langle\psi| \tag{19}$$

is negative definite. Thereby the state $\rho_{out}(\mathbf{m}=D\mathbf{e}_{11})^{(opt)}$ fulfills conditions (18). Due to the covariance condition (4) this non-negativity then implies that also any arbitrary output state $\rho_{out}(\mathbf{m})^{(opt)}$ cannot contain any separable components. Thus, it is optimally entangled.

For the proof of this statement we start from conditions (18) and Eqs.(14) and (16). According to Eqs.(16) and (17) the condition $p_3=0$ implies $\lambda_{4-}=0$. Furthermore, from the non-negativity of λ_{2-} of Eq.(16) and from Eqs. (17) and (18) we obtain the relations

$$\begin{aligned}
\alpha^{(1)} &= \alpha^{(2)},\\
\beta &= \beta^*,\\
\rho_2^{(opt)} &= \frac{1}{2(D-1)}\sum_{j=2}^{D}\{|1j\rangle\langle 1j|+|j1\rangle\langle j1|-\\
&\quad |1j\rangle\langle j1|-|j1\rangle\langle 1j|\},\\
\rho_4^{(opt)} &= \frac{1}{(D-1)(D-2)}\sum_{2=i<j}^{D}\{|ij\rangle\langle ij|+|ji\rangle\langle ji|-\\
&\quad |ij\rangle\langle ji|-|ji\rangle\langle ij|\}.
\end{aligned} \tag{20}$$

Thus, the parameters of Eqs.(18) imply that the resulting output state

$$\rho_{out}^{(opt)}(\mathbf{m}=D\mathbf{e}_{11})=(1-p_4)\rho_2^{(opt)}\oplus p_4\rho_4^{(opt)} \tag{21}$$

is a convex sum of two-particle quantum states which are anti-symmetric with respect to permutations of both quantum systems. Let us consider now the state ρ' of Eq.(19). For an arbitrary state $|\psi\rangle=|\varphi\rangle\otimes|\chi\rangle$ we can always choose a unitary transformation U in such a way that $\langle 1|U|\varphi\rangle$ and $\langle 1|U|\chi\rangle$ are both non-zero. This unitary transformation may be interpreted passively as a change of basis in the one-particle Hilbert spaces. Applying the same unitary transformation to state $\rho_{out}^{(opt)}(\mathbf{m}=D\mathbf{e}_{11})$

produces a convex sum of anti-symmetric two-particle states so that $\langle 11|U\otimes U\rho_{out}^{(opt)}U^\dagger\otimes U^\dagger|11\rangle = 0$. Thus, assuming the existence of a state $|\psi\rangle = |\varphi\rangle\otimes|\chi\rangle$ and a probability $\lambda > 0$ implies that for this particular unitary transformation U the diagonal density matrix element $\langle 11|U\otimes U\rho' U^\dagger\otimes U^\dagger|11\rangle = 0 - \lambda\langle 1|U|\varphi\rangle\langle 1|U|\chi\rangle$ is negative. Therefore ρ' is negative definite for any choice of the states $|\varphi\rangle$ and $|\chi\rangle$ and for any $\lambda > 0$. Therefore a non-zero value of λ is not possible in Eq.(19). So we conclude that the two-particle state of Eq.(21) is optimally entangled. By covariance the same property applies to all possible output states. This completes our proof.

B. Basic properties of optimally entangled output states

Thus, the parameters

$$\begin{aligned} 0 \leq\ & p_4 \leq 1, \\ p_1 &= 0, \\ p_3 &= 0, \\ \alpha^{(1)} &= \alpha^{(2)}, \\ \beta &= \beta^* \end{aligned} \tag{22}$$

characterize all possible universal quantum processes which produce optimally entangled two-particle output states. The resulting output states are statistical mixtures of anti-symmetric states. Explicitly they are given by Eqs.(20), (21) and by applying the covariance condition (4). In addition they exhibit other noteworthy properties which will be discussed in the following.

The partial transpose of the output state $\rho_{out}^{(opt)}(\mathbf{m} = D\mathbf{e}_{11})$ of Eq.(21) has always a negative eigenvalue of magnitude

$$\begin{aligned}\Lambda = &-\frac{p_4}{2(D-1)} - \{\frac{p_4}{(D-1)}^2 + \\ &4(D-1)[\frac{1}{2} - p_3\frac{D-2}{2(D-1)} - \frac{p_4}{2} - p_2\frac{D}{2(D-1)}]^2\}^{1/2}.\end{aligned} \tag{23}$$

Therefore, by covariance all optimally entangled states which are produced by these universal entanglement processes are freely entangled [6].

Due to covariance all output states resulting from the same universal optimal entanglement process have the same von Neumann entropy of magnitude

$$S(p_4) = p_4\ln\frac{(D-1)(D-2)}{2p_4} + (1-p_4)\ln\frac{(D-1)}{1-p_4}. \tag{24}$$

Thus, for $D > 4$ the universal entanglement process with $p_4 = 0$ produces output states with the smallest possible von Neumann entropy, namely

$$S_{min} \equiv S(p_4 = 0) = \ln(D-1). \tag{25}$$

For $D < 4$ this process of minimal von Neumann entropy is characterized by $p_4 = 1$ and the corresponding minimal entropy is given by

$$S_{min} \equiv S(p_4 = 1) = \ln\frac{(D-1)(D-2)}{2}. \tag{26}$$

For $D = 3$ the resulting pure anti-symmetric output state is orthogonal to the pure input state. Geometrically this anti-symmetric output state may be viewed as representing the unique plane which is orthogonal to the input state. This way this process preserves information about the input state. For $D = 4$ both processes, i.e. $p_4 = 0$ and $p_4 = 1$, yield the same von Neumann entropy for the output states. As apparent from Fig.3 this possibility of a 'coexistence' of two universal optimal entanglement processes for $D = 4$ resembles some of the signatures of a second order phase transition. Within the one-parameter family of optimal universal entanglement processes the process characterized by $p_4 = (D-2)/D$ (or equivalently $C = -1/[D(D-1)]$) gives rise to output states with the largest possible value of the von Neumann entropy, namely

$$S_{max} \equiv S(p_4 = \frac{(D-2)}{D}) = \ln\frac{D(D-1)}{2}. \tag{27}$$

Thus this process generates an output state which is a maximal mixture of all possible $(D-1)(D-2)/2$ anti-symmetric two-particle states.

Let us finally, determine the index of correlation [7]

$$IC(\rho) = S(R_1(\mathbf{m})) + S(R_2(\mathbf{m})) - S(\rho_{out}(\mathbf{m})) \tag{28}$$

of the one-parameter family of optimal entanglement processes. Thereby

$$\begin{aligned} R_1(\mathbf{m}) &\equiv \mathrm{Tr}_2\{\rho_{out}(\mathbf{m})\}, \\ R_2(\mathbf{m}) &\equiv \mathrm{Tr}_1\{\rho_{out}(\mathbf{m})\} \end{aligned} \tag{29}$$

denote the reduced density operators of the first and second quantum system. This index of correlation or mutual entropy serves as a measure for the classical and quantum correlations between both quantum systems. For a given value of p_4 the corresponding index of correlation is given by

$$IC(p_4) = \ln\frac{4}{1+p_4} + p_4\ln\frac{2p_4(D-1)}{(1+p_4)(D-2)}. \tag{30}$$

From this relation it is apparent that $IC(p_4)$ has a local minimum for $p_4 = (D-2)/D$. Thus, the optimal entanglement process with the largest possible von Neumann entropy produces output states with the smallest possible mutual entropy. Furthermore, the output states of the optimal entanglement process with $p_4 = 0$ have the largest possible index of correlation, i.e. $IC(p_4 = 0) = 2\ln 2$. It is remarkable that this latter index of correlation is indendent of the dimension of the Hilbert spaces

D and that this value is equal to the mutual entropy of a Bell state.

It is also of interest to investigate to which extent the optimally entangled output states preserve information about the initial pure input state $\rho_{in}(\mathbf{m})$. This information about the input state is characterized by the generalized Bloch vector $\mathbf{m}$. In the output state of Eq.(10) this information is contained in the terms proportional to the parameters $\alpha^{(1)}$, $\alpha^{(2)}$ and β. The parameters $\alpha^{(1)}$ and $\alpha^{(2)}$ characterize the information about the initial pure input state which is still contained in the two-particle output state in each subsystem separately, i.e. in the reduced states

$$R_1(\mathbf{m}) = \frac{\mathbf{1}}{D} + D\alpha^{(1)} m_{ij} \mathbf{A}_{ij},$$
$$R_2(\mathbf{m}) = \frac{\mathbf{1}}{D} + D\alpha^{(2)} m_{ij} \mathbf{A}_{ij} \quad (31)$$

of the first and second quantum system. The parameter β characterizes the information about the input state which is distributed over both quantum systems. This latter property is apparent from the fact that this parameter appears in Eq.(10) with tensor products of the form $\mathbf{A}_{ij} \otimes \mathbf{A}_{jl}$ and $\mathbf{A}_{ji} \otimes \mathbf{A}_{lj}$. According to Eqs.(17) for a given value of p_4 these characteristic quantities are given by

$$\alpha^{(1)} + \alpha^{(2)} = \frac{(D-2)}{2D^2(D-1)} - \frac{p_4}{2D(D-1)},$$
$$\beta + \beta^* = -\frac{1}{D(D-1)} + \frac{p_4}{(D-2)(D-1)}. \quad (32)$$

Thus, the universal entanglement process with $p_4 = 0$ yields

$$\alpha^{(1)}_{max} = (D-2)/[4D^2(D-1)] \quad (33)$$

and preserves the maximum amount of information about the initial state in each subsystem separately. It is instructive to compare this maximum value for $\alpha^{(1)}_{max}$ with the corresponding value achievable by an optimal quantum cloning process which maximizes $\alpha^{(1)}$ with respect to all possible universal quantum processes. Its optimal value is given by $\alpha^{(1)}_{clone} = (D-2)/[4D^2(D-1)] + 1/[2D^2(D-1)(D+1)]$. Thus, for $D > 2$ both values differ by terms of relative magnitude $O(1/D^2)$ so that their difference tends to zero rapidly with increasing dimension D of the one-particle Hilbert spaces. This demonstrates that for $D \gg 2$ an optimal universal entanglement process with $p_4 = 0$ preserves almost as much information about the orientation of the initial quantum state as an optimal universal cloning process (compare with Fig.4).

The optimal entanglement process with $p_4 = (D-2)/D$ yields $\alpha^{(1)} = \alpha^{(2)} = \beta = 0$ so that all information about the orientation of the initial quantum state $\rho_{in}(\mathbf{m})$ is lost. Its resulting output states are independent of the input states and they are scalars with respect to unitary transformations of the form $U \otimes U$ and with respect to permutations between both particles. This particular process is the only one within the one-parameter family of optimal universal entanglement processes which fulfills the additional requirement

$$R_1(\mathbf{m}) = R_2(\mathbf{m}) = \mathbf{1}/D. \quad (34)$$

Though this property is characteristic for all Bell states it does not hold for the output states which are generated by all other optimal universal entanglement processes with $p_4 \neq (D-2)/D$ which are possible for $D > 2$.

Let us finally comment on the special case of qubits, i.e. $D = 2$, for which some of the considerations of this chapter have to be modified. In this case ρ_4 disappears from Eq.(21). Consequently only one optimal entanglement process is possible which is characterized by $p_1 = p_3 = 0$ and by the anti-symmetric output state

$$\rho^{(opt)}_{out}(\mathbf{m} \equiv D\mathbf{e}_{11}) = \rho^{(opt)}_2. \quad (35)$$

Thus, in this case the one-parameter family of optimal universal entanglement processes which is possible for $D > 2$ collapses to a single process whose output states are independent of the input states.

IV. CONCLUSIONS

It has been demonstrated that in Hilbert spaces of dimensions larger than two the linear character of quantum mechanics is compatible with the existence of optimal universal two-particle entanglement processes which preserve information about initial input states. These optimal universal entanglement processes form a one-parameter family and their resulting output states are statistical mixtures of anti-symmetric states. This situation is completely different from the case of qubits where only one optimal universal two-particle entanglement process is possible which destroys all information about any input state. In higher dimensional Hilbert spaces there is only one particular member of this one-parameter family for which all information about an initial input state is lost completely during the entanglement process. All other processes within this family preserve this information at least partly. The degree of preservation of this information about the input state approaches the degree which is achievable by optimal universal cloning processes. The dimensional dependence of the optimal universal entanglement process whose output states have minimal von Neumann entropy exhibits signatures of a second order 'phase transition' for Hilbert space dimensions equal to four. For this particular dimension two different optimal universal entanglement processes are possible which produce output states with minimal von Neumann entropy.

The presented investigations indicate that convex sums of anti-symmetric quantum states might also play a predominant role in universal entanglement processes which

involve more than two quantum systems. Furthermore, entanglement processes which also preserve information about an input state might have interesting applications in various branches of quantum information processing, such as quantum cryptography or quantum error correction. Thus, the presented results indicate that further exploration of quantum information processing beyond qubits may offer unexpected and useful surprises.

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$$\begin{tikzcd}
\mathbf{m}_0 \arrow[r] \arrow[d, "\mathcal{P}"] & \mathbf{m} = \mathbf{U}\mathbf{m}_0 \arrow[d, "\mathcal{P}"] \\
\rho_{out}(\mathbf{m}_0) \arrow[r] & \rho_{out}(\mathbf{m}) = \mathbf{U}\otimes\mathbf{U}\rho_{out}(\mathbf{m}_0)\mathbf{U}^\dagger\otimes\mathbf{U}^\dagger
\end{tikzcd}$$

FIG. 1. Pictorial representation of the symmetry (covariance) condition which characterizes universal quantum processes

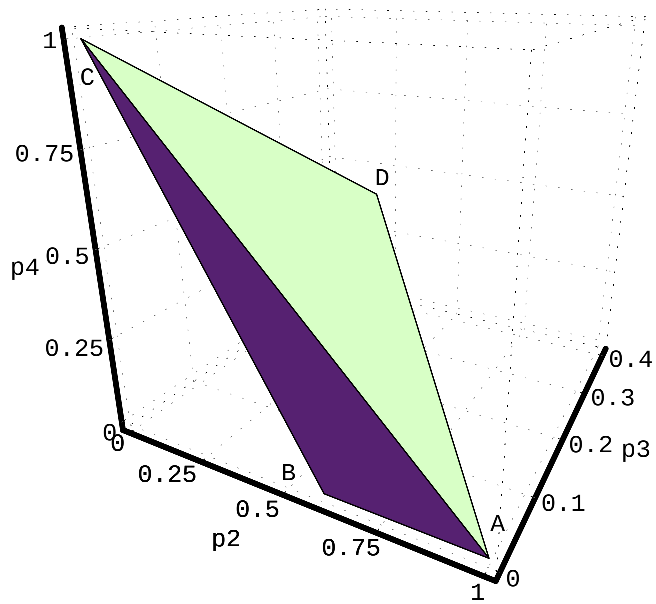

FIG. 2. Convex set of points (p_2, p_3, p_4) characterizing all possible universal quantum processes for $\alpha^{(1)} = \alpha^{(2)}$, $\beta = \beta^*$ and $D = 4$.

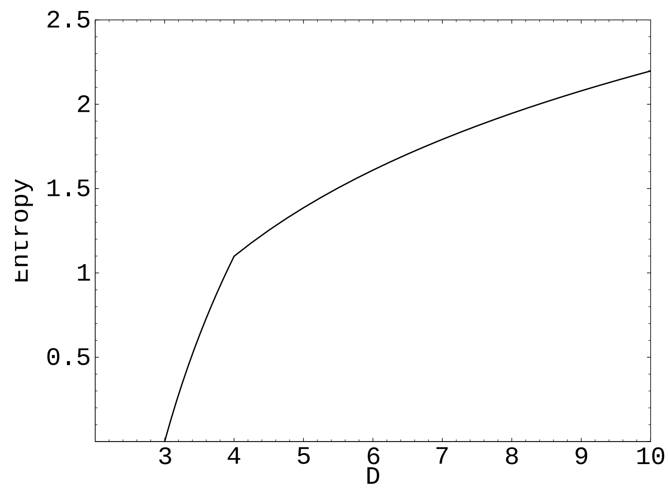

FIG. 3. Minimal values of the von Neumann entropy of optimal universal entanglement processes (compare with Eqs.(25) and (26)) as a function of D.

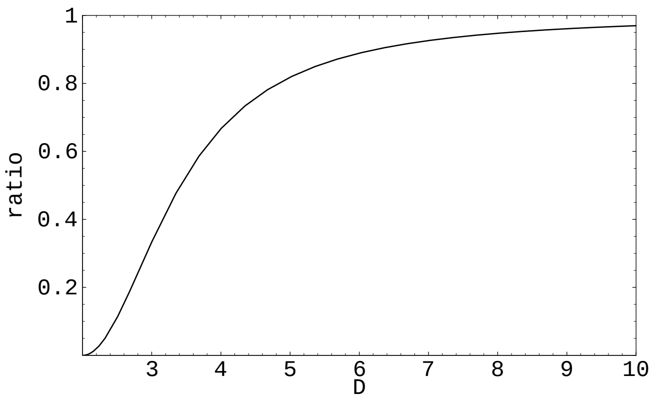

FIG. 4. Dimensional dependence of the ratio between $\alpha_{max}^{(1)}$ as defined by Eq.(33) and the corresponding value $\alpha_{clone}^{(1)}$ characterizing the optimal universal cloning process. It is for $D = 2$ only that in the optimal universal entanglement process all information about any input state is lost.